

WEAK DISTRIBUTIVITY IMPLYING DISTRIBUTIVITY

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ABSTRACT. Let \mathbb{B} be a complete Boolean algebra. We show that if λ is an infinite cardinal and \mathbb{B} is weakly (λ^ω, ω) -distributive, then \mathbb{B} is $(\lambda, 2)$ -distributive. Using a similar argument, we show that if κ is a weakly compact cardinal such that \mathbb{B} is weakly $(2^\kappa, \kappa)$ -distributive and \mathbb{B} is $(\alpha, 2)$ -distributive for each $\alpha < \kappa$, then \mathbb{B} is $(\kappa, 2)$ -distributive.

1. INTRODUCTION

Given sets A and B , AB denotes the set of functions from A to B . In this article, λ and κ will denote ordinals, although usually they can be assumed to be infinite cardinals. As defined in [6], given λ and κ , we say that a complete Boolean algebra \mathbb{B} is (λ, κ) -*distributive* iff

$$\prod_{\alpha < \lambda} \sum_{\beta < \kappa} u_{\alpha, \beta} = \sum_{f: \lambda \rightarrow \kappa} \prod_{\alpha < \lambda} u_{\alpha, f(\alpha)}$$

for any $\langle u_{\alpha, \beta} \in \mathbb{B} : \alpha < \lambda, \beta < \kappa \rangle$. Given maximal antichains $A_1, A_2 \subseteq \mathbb{B}$, we say that A_2 *refines* A_1 iff $(\forall a_2 \in A_2)(\exists a_1 \in A_1) a_2 \leq_{\mathbb{B}} a_1$. It is a fact that \mathbb{B} is (λ, κ) -distributive iff each size λ collection of maximal antichains in \mathbb{B} each of size κ has a common refinement. There is also a useful characterization in terms of forcing (which can be found in [6] as Theorem 15.38):

Fact 1.1. *A complete Boolean algebra \mathbb{B} is (λ, κ) -distributive iff*

$$1 \Vdash_{\mathbb{B}} (\forall f: \check{\lambda} \rightarrow \check{\kappa}) f \in \check{V}.$$

Unfortunately, the definition of weakly distributive varies in the literature (for example [7]). We will use the one given by Jech (see [6]). That is, we say that a complete Boolean algebra \mathbb{B} is *weakly* (λ, κ) -*distributive* iff

$$\prod_{\alpha < \lambda} \sum_{\beta < \kappa} u_{\alpha, \beta} = \sum_{g: \lambda \rightarrow \kappa} \prod_{\alpha < \lambda} \sum_{\beta < g(\alpha)} u_{\alpha, \beta}.$$

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This definition has a natural characterization in terms of forcing. Given a set X and $f, g : X \rightarrow \kappa$, we write $f \leq g$ iff g everywhere dominates f . That is,

$$(\forall x \in X) f(x) \leq g(x).$$

Fact 1.2. *A complete Boolean algebra \mathbb{B} is weakly (λ, κ) -distributive iff*

$$1 \Vdash_{\mathbb{B}} (\forall f : \check{\lambda} \rightarrow \check{\kappa}) (\exists g : \check{\lambda} \rightarrow \check{\kappa}) g \in \check{V} \wedge f \leq g.$$

We will show the following:

Theorem (A). *Let λ be an infinite cardinal. If*

- 1) \mathbb{B} is weakly (λ^ω, ω) -distributive,

then \mathbb{B} is $(\lambda, 2)$ -distributive.

Theorem (B). *Let κ be a weakly compact cardinal. If*

- 1) \mathbb{B} is weakly $(2^\kappa, \kappa)$ -distributive and
- 2) \mathbb{B} is $(\alpha, 2)$ -distributive for each $\alpha < \kappa$,

then \mathbb{B} is $(\kappa, 2)$ -distributive.

We will then discuss why Theorem (B) does not hold when we have $\kappa = \omega_1$ instead of κ being weakly compact, and we will show one way to fix the situation using the tower number. Finally, we use the same idea using the tower number to prove a variation of Theorem (A) involving weak (λ^κ, κ) -distributivity for $\kappa > \omega$.

2. FUNCTIONS FROM λ^ω TO ω

The proof of the following lemma uses the fact that well-foundedness of trees is absolute. It is crucial, for what follows, that this lemma does not require ${}^\omega\lambda \subseteq M$. See [4] for motivation and discussion.

Lemma 2.1. *For each $A \subseteq \lambda$, there is a function $f : {}^\omega\lambda \rightarrow \omega$ such that whenever M is a transitive model of ZF such that $\lambda \in M$ and some $g : ({}^\omega\lambda)^M \rightarrow \omega$ in M satisfies*

$$(\forall x \in ({}^\omega\lambda)^M) f(x) \leq g(x),$$

then $A \in M$.

Proof. Fix $A \subseteq \lambda$. Define $f : {}^\omega\lambda \rightarrow \omega$ by

$$f(x) := \begin{cases} 0 & \text{if } (\forall n < \omega) x(n) \notin A, \\ n+1 & \text{if } x(n) \in A \text{ but } (\forall m < n) x(m) \notin A. \end{cases}$$

Let M be a transitive model of ZF such that $\lambda \in M$ but $A \notin M$. Suppose, towards a contradiction, that there is some $g \in M$ satisfying $(\forall x \in (\omega\lambda)^M) f(x) \leq g(x)$. Let B be the set

$$B := \{t \in {}^{<\omega}\lambda : g(x) \geq |t| \text{ for all } x \text{ in } M \text{ extending } t\}.$$

Notice that $B \in M$. Let $T \subseteq {}^{<\omega}\lambda$ be the set of elements of B all of whose initial segments are also in B . Note that T is a tree and $T \in M$.

For all $a \in \lambda$, $a \in A$ implies $\langle a \rangle \in B$. Thus, there must be some $a_0 \in \lambda$ such that $a_0 \notin A$ but $\langle a_0 \rangle \in B$. If there was not, then A could be defined in M by $A = \{a \in \lambda : \langle a \rangle \in B\}$.

Next, for all $a \in \lambda$, $a \in A$ implies $\langle a_0, a \rangle \in B$. Thus, by similar reasoning as before, there must be some $a_1 \in \lambda$ such that $a_1 \notin A$ but $\langle a_0, a_1 \rangle \in B$. Continuing like this, we can construct a sequence $x \in {}^\omega\lambda$ satisfying $(\forall n < \omega) x \restriction n \in B$. Thus, $(\forall n < \omega) x \restriction n \in T$, so T is not well-founded.

Since well-foundedness is absolute, there is some path x' through T in M . Since $(\forall n < \omega) x' \restriction n \in B$, we have $(\forall n < \omega) g(x') \geq n$, which is impossible. \square

This implies the following lemma, whose order of quantifiers is not as powerful, but the functions have the ordinal $(\lambda^\omega)^M$ instead of the set of sequences $(\omega\lambda)^M$ as their domains:

Lemma 2.2. *Let M be a transitive model of ZF such that the ordinal λ is in M and $(\omega\lambda)^M$ can be well-ordered in M . Assume that for each $f : (\lambda^\omega)^M \rightarrow \omega$ there is some $g : (\lambda^\omega)^M \rightarrow \omega$ in M such that $f \leq g$. Then $\mathcal{P}(\lambda) \subseteq M$.*

Proof. Consider any $A \in \mathcal{P}(\lambda)$. Use the lemma above with A to get $\tilde{f} : {}^\omega\lambda \rightarrow \omega$ such that if $\tilde{g} : (\omega\lambda)^M \rightarrow \omega$ is any function in M which satisfies

$$(1) \quad (\forall x \in (\omega\lambda)^M) \tilde{f}(x) \leq \tilde{g}(x),$$

then $A \in M$. Since $(\omega\lambda)^M$ can be well-ordered in M , fix a bijection

$$\eta : (\lambda^\omega)^M \rightarrow (\omega\lambda)^M$$

in M . Define $f : (\lambda^\omega)^M \rightarrow \omega$ by

$$f(\alpha) := \tilde{f}(\eta(\alpha)).$$

That is, the following diagram commutes:

$$\begin{array}{ccc} (\omega\lambda)^M & \xrightarrow{\tilde{f}} & \omega \\ \eta \uparrow & \nearrow f & \\ (\lambda^\omega)^M & & \end{array}$$

By hypothesis, let $g : (\lambda^\omega)^M \rightarrow \omega$ be a function in M which everywhere dominates f . Define $\tilde{g} : (\omega\lambda)^M \rightarrow \omega$ by

$$\tilde{g}(x) := g(\eta^{-1}(x)).$$

We have that $\tilde{g} \in M$ and \tilde{g} satisfies 1, so by the hypothesis on \tilde{f} , $A \in M$. \square

We now have the main result of this section:

Theorem (A). *Let \mathbb{B} be a complete Boolean algebra and λ be an infinite cardinal. If \mathbb{B} is weakly (λ^ω, ω) -distributive, then \mathbb{B} is $(\lambda, 2)$ -distributive.*

Proof. Let $\mu := \lambda^\omega$. Assume \mathbb{B} is weakly (μ, ω) -distributive. Force with \mathbb{B} . Every $f : \mu \rightarrow \omega$ in the extension can be everywhere dominated by some $g : \mu \rightarrow \omega$ in the ground model, so applying the lemma above in the extension (setting M to be the ground model) tells us that the $\mathcal{P}(\lambda)$ of the extension is included in the ground model. Hence, \mathbb{B} is $(\lambda, 2)$ -distributive. \square

3. FUNCTIONS FROM 2^κ TO κ WITH κ WEAKLY COMPACT

The first lemma in the previous section was the key to the theorem there. We have a parallel lemma here which, instead of using the absoluteness of trees being well-founded, uses the tree property to get similar absoluteness. It is important that this lemma does not require ${}^\kappa 2 \subseteq M$. By weakly compact, we mean strongly inaccessible and having the tree property.

Lemma 3.1. *For each $a \in {}^\kappa 2$, there is a function $f : {}^\kappa 2 \rightarrow \kappa$ such that whenever M is a transitive model of ZF such that $\kappa \in M$, ${}^{<\kappa} 2 \subseteq M$, $({}^\kappa 2)^M$ can be well-ordered in M , $(\kappa \text{ is weakly compact})^M$, and some $g : ({}^\kappa 2)^M \rightarrow \kappa$ in M satisfies*

$$(\forall x \in ({}^\kappa 2)^M) f(x) \leq g(x),$$

then $a \in M$.

Proof. Fix $a \in {}^\kappa 2$. Let $f : {}^\kappa 2 \rightarrow \kappa$ be the function

$$f(x) := \begin{cases} 0 & \text{if } (\forall \alpha < \kappa) x(\alpha) = a(\alpha), \\ \alpha + 1 & \text{if } x(\alpha) \neq a(\alpha) \text{ but } (\forall \beta < \alpha) x(\beta) = a(\beta). \end{cases}$$

Let M be an appropriate transitive model of ZF. Suppose $g : ({}^\kappa 2)^M \rightarrow \kappa$ in M satisfies $(\forall x \in ({}^\kappa 2)^M) f(x) \leq g(x)$. We will show that $a \in M$.

Suppose, towards a contradiction, that $a \notin M$. Let

$$B := \{t \in {}^{<\kappa} 2 : g(x) \geq \text{Dom}(t) \text{ for all } x \text{ in } M \text{ extending } t\}.$$

Note that by definition, there cannot be any $x \in {}^\kappa 2$ in M satisfying $(\forall \alpha < \kappa) x \restriction \alpha \in B$ because if there was such an x , we would have $(\forall \alpha < \kappa) g(x) \geq \alpha$, which is impossible. Since B need not be a tree, let $T \subseteq {}^{<\kappa} 2$ be the tree of those elements of B all of whose initial segments are also in B . Again, T cannot have a length κ path in M . Note that for each $\alpha < \kappa$, $a \restriction \alpha \in B$. This is because any $x \in {}^\kappa 2$ in M which extends $a \restriction \alpha$ differs from a (since $a \notin M$), and the smallest γ such that $x(\gamma) \neq a(\gamma)$ must be $\geq \alpha$, so

$$g(x) \geq f(x) = \gamma + 1 > \gamma \geq \alpha = \text{Dom}(a \restriction \alpha).$$

Since $(\forall \alpha < \kappa) a \restriction \alpha \in B$, also $(\forall \alpha < \kappa) a \restriction \alpha \in T$.

Now, $B \in M$ (since ${}^{<\kappa} 2 \subseteq M$ and $g \in M$) and so $T \in M$. Since $(\forall \alpha < \kappa) a \restriction \alpha \in T$, $(T \text{ has height } \kappa)^M$. Since $(\kappa \text{ is strongly inaccessible})^M$, we have $(T \text{ is a } \kappa\text{-tree})^M$. Since $(\kappa \text{ has the tree property})^M$, there is a length κ path through T in M , which we said earlier was impossible. \square

As before, this implies the following lemma, whose order of quantifiers is not as powerful, but the functions have the ordinal $(2^\kappa)^M$ instead of the set of sequences $({}^\kappa 2)^M$ as their domains:

Lemma 3.2. *Let M be a transitive model of ZF such that the ordinal κ is in M , ${}^{<\kappa} 2 \subseteq M$, $({}^\kappa 2)^M$ can be well-ordered in M , and $(\kappa \text{ is weakly compact})^M$. Assume that for each $f : (2^\kappa)^M \rightarrow \kappa$ there is some $g : (2^\kappa)^M \rightarrow \kappa$ in M such that $f \leq g$. Then $\mathcal{P}(\kappa) \subseteq M$.*

Proof. The proof is similar to that of Lemma 2.2. \square

As before, the main result of this section follows:

Theorem (B). *Let \mathbb{B} be a complete Boolean algebra and κ be a weakly compact cardinal. If \mathbb{B} is weakly $(2^\kappa, \kappa)$ -distributive and \mathbb{B} is $(\alpha, 2)$ -distributive for each $\alpha < \kappa$, then \mathbb{B} is $(\kappa, 2)$ -distributive.*

Proof. This follows from the lemma above just as Theorem (A) followed from Lemma 2.2. \square

4. THE TOWER NUMBER

One might hope that Theorem (B) holds when $\kappa = \omega_1$ instead of κ being weakly compact. That is, one might hope that if a complete Boolean algebra \mathbb{B} is weakly $(2^{\omega_1}, \omega_1)$ -distributive and $(\omega, 2)$ -distributive, then it is $(\omega_1, 2)$ -distributive. Unfortunately, this cannot be proved in ZFC because \mathbb{B} could be a Suslin algebra (a Suslin algebra is c.c.c. and therefore is weakly (λ, ω_1) -distributive for any λ). However, if we add the assumption that $1 \Vdash_{\mathbb{B}} (\omega_1 < \mathfrak{t})$, where we will define \mathfrak{t} soon, then \mathbb{B} is $(\omega_1, 2)$ -distributive. The argument is simpler than that of Theorem (A) and Theorem (B) and does not need the hypothesis of weak $(2^{\omega_1}, \omega_1)$ -distributivity. As a final twist, we will combine several ideas to prove a variation of Theorem (A).

Recall that \mathfrak{t} , the *tower number*, is the smallest length of a sequence

$$\langle S_\alpha \in [\omega]^\omega : \alpha < \kappa \rangle$$

satisfying $(\forall \alpha < \beta < \kappa) S_\alpha \supseteq^* S_\beta$ but there is no $S \in [\omega]^\omega$ satisfying $(\forall \alpha < \kappa) S_\alpha \supseteq^* S$ (where $S_1 \subseteq^* S_2$ means $S_1 - S_2$ is finite). It is not hard to see that $\omega_1 \leq \mathfrak{t} \leq 2^\omega$. See [1] for more on \mathfrak{t} and related cardinals. The following lemma is the key. The idea is borrowed from Farah in [3], who got the idea from Dordal in [2], who got the idea from Booth.

Lemma 4.1. *Let κ be such that $\omega_1 \leq \kappa < \mathfrak{t}$. Let M be a transitive model of ZFC such that $\kappa \in M$ and $(\forall \alpha < \kappa) \mathcal{P}(\alpha) \subseteq M$. Then $\mathcal{P}(\kappa) \subseteq M$.*

Proof. Fix κ and M . Since $\kappa \in M$ and $(\forall \alpha < \kappa) \mathcal{P}(\alpha) \subseteq M$, we have ${}^{<\kappa}2 \subseteq M$. Let $F : {}^{<\kappa}2 \rightarrow [\omega]^\omega$ be a function in M such that for all $t_1, t_2 \in {}^{<\kappa}2$,

- 1) $t_1 \sqsubseteq t_2 \Rightarrow F(t_1) \supseteq^* F(t_2)$, and
- 2) $t_1 \perp t_2 \Rightarrow F(t_1) \cap F(t_2)$ is finite.

Such functions are easy to construct by induction (and the Axiom of Choice). The construction will not get stuck at a limit stage $\gamma < \kappa$ because given $t \in {}^\gamma 2 \subseteq M$ and $\langle F(t \restriction \alpha) : \alpha < \gamma \rangle$, since $\gamma < \mathfrak{t}$ there is some $S \in [\omega]^\omega \subseteq M$ such that $(\forall \alpha < \gamma) S \subseteq^* F(t \restriction \alpha)$. The set $F(t)$ can be defined to be the least such S according to some fixed well-ordering of $[\omega]^\omega$.

Now, consider any $a \in {}^\kappa 2$. We will show that $a \in M$. The sequence $\langle F(a \restriction \alpha) : \alpha < \kappa \rangle$ is a \supseteq^* -chain (in V) of length κ . Since $\kappa < \mathfrak{t}$, fix some $S \in [\omega]^\omega$ satisfying

$$(\forall \alpha < \kappa) S \subseteq^* F(a \restriction \alpha).$$

Since $\mathcal{P}(\omega) \subseteq M$, in particular $S \in M$. Within M , the function F and the set S can be used together to define a :

$$a = \bigcup \{t \in {}^{<\kappa}2 : S \subseteq^* F(t)\}. \quad \square$$

By applying the lemma above inductively, we get an improvement:

Lemma 4.2. *Let κ be such that $\omega_1 \leq \kappa < \mathfrak{t}$. Let M be a transitive model of ZFC such that $\mathcal{P}(\omega) \subseteq M$. Then $\mathcal{P}(\kappa) \subseteq M$.*

This last lemma is closely related to the fact that $2^\kappa = 2^\omega$ whenever $\kappa < \mathfrak{t}$. A proof of this using an argument similar to Lemma 4.1 can be found in [1]. Martin's Axiom (MA) implies $\mathfrak{t} = 2^\omega$, but the original proof [8] that MA implies $2^\kappa = 2^\omega$ whenever $\kappa < 2^\omega$ used the almost disjoint coding poset. We now have the application to complete Boolean algebras:

Proposition 4.3. *Let κ be an infinite cardinal. Let \mathbb{B} be a complete Boolean algebra such that \mathbb{B} is $(\omega, 2)$ -distributive and $1 \Vdash_{\mathbb{B}} (\check{\kappa} < \mathfrak{t})$. Then \mathbb{B} is $(\kappa, 2)$ -distributive.*

Proof. Apply Lemma 4.2 in the forcing extension with M equal to the ground model. \square

Let κ be such that $\omega_1 \leq \kappa < \mathfrak{t}$. Any $A \in [\omega]^\omega$ can be partitioned into 2^ω infinite sets with pairwise finite intersection. Thus, fixing $\lambda \leq 2^\omega$, the function $F : {}^{<\kappa}2 \rightarrow [\omega]^\omega$ in Lemma 4.1 can be replaced by a function $F : {}^{<\kappa}\lambda \rightarrow [\omega]^\omega$ satisfying the same conditions. Slightly modifying the proof of Lemma 4.1, we get that if M is a transitive model of ZFC such that $\lambda \in M$ and $(\forall \alpha < \kappa) {}^\alpha\lambda \subseteq M$, then ${}^\kappa\lambda \subseteq M$. Inductively applying this fact yields an improvement:

Lemma 4.4. *Let κ and λ be such that $\omega_1 \leq \kappa < \mathfrak{t}$ and $\lambda \leq 2^\omega$. Let M be a transitive model of ZFC such that $\lambda \in M$ and ${}^\omega\lambda \subseteq M$. Then ${}^\kappa\lambda \subseteq M$.*

Now we may combine Lemma 4.4 with the argument in Lemma 2.1. The case $\kappa = \omega$ of this next lemma is already handled by Lemma 2.1.

Lemma 4.5. *Let κ and λ be such that $\omega \leq \kappa < \mathfrak{t}$ and $\lambda \leq 2^\omega$. For each $A \subseteq \lambda$, there is a function $f : {}^\kappa\lambda \rightarrow \kappa$ such that whenever M is a transitive model of ZFC such that ${}^\omega\lambda \subseteq M$ (and therefore ${}^\kappa\lambda \subseteq M$) and some $g : {}^\kappa\lambda \rightarrow \kappa$ in M satisfies $f \leq g$, then $A \in M$.*

Proof. Fix κ , λ , and A . Define $f : {}^\kappa\lambda \rightarrow \kappa$ by

$$f(x) := \begin{cases} 0 & \text{if } (\forall \alpha < \kappa) x(\alpha) \notin A, \\ \alpha + 1 & \text{if } x(\alpha) \in A \text{ but } (\forall \beta < \alpha) x(\beta) \notin A. \end{cases}$$

This is the analogue of the function f defined in Lemma 2.1. Now fix M and some $g : {}^\kappa\lambda \rightarrow \kappa$ in M satisfying $f \leq g$. Note that $2^\omega \in M$ so therefore $\kappa, \lambda \in M$. Let

$$B := \{t \in {}^{<\kappa}\lambda : g(x) \geq \text{Dom}(t) \text{ for all } x \text{ extending } t\}.$$

Since ${}^{<\kappa}\lambda \cup \{\kappa, \lambda, g\} \subseteq M$, also $B \in M$.

Assume towards a contradiction, that $A \notin M$. Arguing just as in Lemma 2.1, there is some $x \in {}^\kappa\lambda$ satisfying $(\forall \alpha < \kappa) x \restriction \alpha \in B$. Since ${}^\kappa\lambda \subseteq M$, we have $x \in M$, and in particular x is in the domain of g . We now have $(\forall \alpha < \kappa) g(x) \geq \alpha$, which is impossible. \square

Lemma 4.6. *Let κ and λ be such that $\omega \leq \kappa < \mathfrak{t}$ and $\lambda \leq 2^\omega$. Let M be a transitive model of ZFC such that ${}^\omega\lambda \subseteq M$ (and therefore ${}^\kappa\lambda \subseteq M$). Assume that for each $f : (\lambda^\kappa)^M \rightarrow \kappa$ there is some $g : (\lambda^\kappa)^M \rightarrow \kappa$ in M satisfying $f \leq g$. Then $\mathcal{P}(\lambda) \subseteq M$.*

Proof. This follows immediately from the previous lemma. \square

Now follows the theorem:

Theorem 4.7. *Let \mathbb{B} be a complete Boolean algebra. Let κ and λ be such that $1 \Vdash_{\mathbb{B}} (\check{\kappa} < \mathfrak{t})$ and $1 \Vdash_{\mathbb{B}} (\check{\lambda} \leq 2^\omega)$. Assume that \mathbb{B} is (ω, λ) -distributive and weakly (λ^κ, κ) -distributive. Then \mathbb{B} is $(\lambda, 2)$ -distributive.*

5. SUSLIN ALGEBRAS AND $\text{MA}(\omega_1)$

The theorems in this paper relied on absoluteness results concerning trees. We can get a counterexample to a generalization of these theorems by using a Suslin tree (a tree of height ω_1 such that every branch and antichain is at most countable). Recall the following definition:

Definition 5.1. A *Suslin algebra* is a complete Boolean algebra that is atomless, (ω, κ) -distributive for each cardinal κ , and c.c.c.

It is a theorem of ZFC that there exists a Suslin algebra iff there exists a Suslin tree. Furthermore, given a Suslin algebra \mathbb{B} , there is a Suslin tree (turned upside down) that completely embeds into \mathbb{B} , so \mathbb{B} is not $(\omega_1, 2)$ -distributive (see [6]). Immediately, we see that Theorem (B) cannot be changed by simply replacing the weakly compact cardinal κ with ω_1 :

Counterexample 5.2. *Let \mathbb{B} be a Suslin algebra. Then \mathbb{B} is weakly $(2^{\omega_1}, \omega_1)$ -distributive and \mathbb{B} is $(\omega, 2)$ -distributive, but \mathbb{B} is not $(\omega_1, 2)$ -distributive.*

Proof. The only claim left to be verified is that \mathbb{B} is weakly $(2^{\omega_1}, \omega_1)$ -distributive. In fact, we will show that \mathbb{B} is weakly (λ, ω_1) -distributive for all λ . To see why, fix λ and fix a \mathbb{B} -name \dot{f} such that

$$1 \Vdash_{\mathbb{B}} \dot{f} : \check{\lambda} \rightarrow \omega_1.$$

Since \mathbb{B} has the c.c.c., there are only countably many possible values for a given term in the forcing language. In particular, for each $\alpha < \lambda$, there are only countably many possible values for $\dot{f}(\check{\alpha})$. For each $\alpha < \lambda$, let $g(\alpha) < \omega_1$ be the supremum of these possible values. We now have

$$1 \Vdash_{\mathbb{B}} (\forall \alpha < \check{\lambda}) \dot{f}(\alpha) \leq \check{g}(\alpha).$$

Since \dot{f} was arbitrary, by Fact 1.2 \mathbb{B} is weakly (λ, ω_1) -distributive. \square

Unfortunately, the counterexample above used a Suslin algebra, which ZFC does not prove exists. In particular, we ask the following:

Question 5.3. Is it consistent with ZFC that every complete Boolean algebra that is both (ω, κ) -distributive for all κ and weakly (λ, ω_1) -distributive for all λ must also be $(\omega_1, 2)$ -distributive?

The intuitive way to try to affirmatively answer the above question is to consider a model of $\text{MA}(\omega_1)$. By Proposition 4.3, we only need to worry about those \mathbb{B} such that $1 \Vdash_{\mathbb{B}} (\omega_1 = \mathfrak{t})$. We present another result which shows we do not need to worry about complete Boolean algebras that satisfy both a strong chain condition and enough weak distributivity laws. The main idea is the following: if we have a size λ collection \mathcal{C} of antichains in \mathbb{B} each of size κ' , then if \mathbb{B} is weakly (λ, κ') -distributive, then there is a maximal antichain $A \subseteq \mathbb{B}$ such that below each $a \in A$, each antichain in \mathcal{C} has $< \kappa'$ non-zero elements. Assuming also that \mathbb{B} is $(\omega, |\mathbb{B}|)$ -distributive, we can repeatedly apply this construction countably many times until we produce a maximal antichain B_ω such that below each $b' \in B_\omega$, each antichain of \mathbb{B} has only countably many non-zero elements. That is, B_ω will witness that \mathbb{B} is “locally c.c.c.”. Then, we use a result of Baumgartner to conclude that since \mathbb{B} is locally c.c.c. and $(\omega, 2)$ -distributive, \mathbb{B} is either $(\omega_1, 2)$ -distributive or a Suslin tree can be embedded into \mathbb{B} . If we assume there are no Suslin trees (which follows from $\text{MA}(\omega_1)$), we get that \mathbb{B} must be $(\omega_1, 2)$ -distributive.

Theorem (D). *Assume there are no Suslin trees. Let \mathbb{B} be a complete Boolean algebra such that \mathbb{B} is $(\omega, |\mathbb{B}|)$ -distributive, \mathbb{B} is κ -c.c. for some $\kappa < \aleph_{\omega_1}$, and $(\forall \text{uncountable } \kappa' < \kappa) \mathbb{B} \text{ is weakly } (|\mathbb{B}|^{\kappa'}, \kappa')\text{-distributive}$. Then \mathbb{B} is $(\omega_1, 2)$ -distributive.*

Proof. We will construct a sequence of maximal antichains

$$\langle B_n \subseteq \mathbb{B} : n \in \omega \rangle$$

such that $B_0 := \{1_{\mathbb{B}}\}$ and $(\forall n < m < \omega) B_m$ refines B_n . Each B_n will have the property that for any maximal antichain A below an element $b \in B_n$, for each $b' \in B_{n+1}$ extending b , A will have $< |A|$ non-zero elements below b' . We will then define the maximal antichain B_ω to refine each B_n , and we will argue that below each $b_\omega \in B_\omega$, \mathbb{B} is c.c.c.

Let $\kappa < \aleph_{\omega_1}$ be the least cardinal such that \mathbb{B} is κ -c.c. Define $B_0 := \{1_{\mathbb{B}}\}$. We will now define a maximal antichain $B_1 \subseteq \mathbb{B}$ (which trivially refines B_0). Every antichain in \mathbb{B} has size $< \kappa$. Consider an uncountable cardinal $\kappa' = \aleph_\alpha < \kappa$. Let $\lambda := |\mathbb{B}|^{\kappa'}$. Let $\langle A_\beta : \beta < \lambda \rangle$ be an enumeration of the maximal antichains in \mathbb{B} of size κ' . For each $\beta < \lambda$, let $\langle a_{\beta,\gamma} : \gamma < \kappa' \rangle$ be an enumeration of the elements of A_β . Let \dot{G} be the canonical name for the generic filter. Fix a name \dot{f} such that $1 \Vdash \dot{f} : \check{\lambda} \rightarrow \check{\kappa}'$ and

$$1 \Vdash (\forall \beta < \check{\lambda}) \check{a}_{\beta, \dot{f}(\beta)} \in \dot{G}.$$

By hypothesis, \mathbb{B} is weakly (λ, κ') -distributive, so there is a maximal antichain $C_{0,\alpha} \subseteq \mathbb{B}$ (which trivially refines B_0) and a name \dot{g} such that $1 \Vdash \dot{g} : \check{\lambda} \rightarrow \check{\kappa}'$ and

$$1 \Vdash (\forall \beta < \check{\lambda}) \dot{f}(\beta) \leq \dot{g}(\beta).$$

Hence,

$$1 \Vdash (\forall \beta < \check{\lambda}) (\forall \gamma < \check{\kappa}') \gamma > \dot{g}(\beta) \Rightarrow \check{a}_{\beta,\gamma} \notin \dot{G}.$$

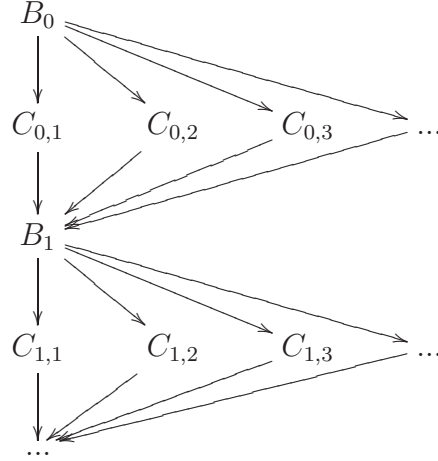
This implies that below each $c \in C_{0,\alpha}$, each A_β has $< |A_\beta| = \kappa'$ non-zero elements. That is, for each $c \in C_{0,\alpha}$ and A_β , there are $< |A_\beta|$ many $a \in A_\beta$ such that $c \wedge a \neq 0_{\mathbb{B}}$.

For each $\aleph_\alpha < \kappa$, we have such a maximal antichain $C_{0,\alpha} \subseteq \mathbb{B}$. Since $\kappa < \aleph_{\omega_1}$, the family $\langle C_{0,\alpha} \subseteq \mathbb{B} : \aleph_\alpha < \kappa \rangle$ is countable. Each $C_{0,\alpha}$ has size $\leq |\mathbb{B}|$, so since \mathbb{B} is $(\omega, |\mathbb{B}|)$ -distributive, we may fix a single maximal antichain $B_1 \subseteq \mathbb{B}$ which refines each $C_{0,\alpha}$. Note that B_1 has the property that for each maximal antichain $A \subseteq \mathbb{B}$ (below $1_{\mathbb{B}}$) and $b' \in B_1$, A has $< |A|$ non-zero elements below b' .

We will now define B_2 . Consider an uncountable cardinal $\kappa' = \aleph_\alpha < \kappa$. Let $\lambda := |\mathbb{B}|^{\kappa'}$. Let $\langle A_\beta : \beta < \lambda \rangle$ be an enumeration of all size κ' antichains that are each a partition of some element of B_1 . Since \mathbb{B} is weakly (λ, κ') -distributive, we may use a similar argument as before to get a maximal antichain $C_{1,\alpha}$ which refines B_1 such that below each $c \in C_{1,\alpha}$, each A_β has $< |A_\beta| = \kappa'$ non-zero elements. This completes the construction of $C_{1,\alpha}$. As before, we may use the $(\omega, |\mathbb{B}|)$ -distributivity of \mathbb{B} to get a common refinement B_2 of every maximal antichain in the

family $\langle C_{1,\alpha} : \aleph_\alpha < \kappa \rangle$. Note that B_2 has the property that for every partition A of some element of B_1 and $b' \in B_2$, A has $< |A|$ non-zero elements below b' .

We may continue this procedure to get a sequence $\langle B_n : n \in \omega \rangle$ of maximal antichains of \mathbb{B} . The following diagram depicts the maximal antichains which we have constructed, where an arrow represents refinement:



Using the $(\omega, |\mathbb{B}|)$ -distributivity of \mathbb{B} once more, we may get a single maximal antichain $B_\omega \subseteq \mathbb{B}$ which refines each B_n . We will now argue that given any maximal antichain $A \subseteq \mathbb{B}$ and $b_\omega \in B_\omega$, A has only countably many non-zero elements below b .

Fix an arbitrary maximal antichain $A_0 \subseteq \mathbb{B}$. Fix $b_\omega \in B_\omega$. Let $\kappa_0 := |A_0|$. If $\kappa_0 \leq \omega$, we are done. If not, let b_1 be the unique element of B_1 above b_ω . By the construction of B_1 , A_0 has $< \kappa_0$ non-zero elements below b_1 . Let $\kappa_1 < \kappa_0$ be the number of such non-zero elements. That is, letting

$$A_1 := \{a \wedge b_1 : a \in A_0\},$$

we have $|A_1| = \kappa_1 < \kappa_0$. If $\kappa_1 \leq \omega$, we are done because $|\{a \wedge b_\omega : a \in A_0\}| \leq |A_1| \leq \omega$. Otherwise, let b_2 be the unique element of B_2 above b_ω . By the construction of B_2 , A_1 has $< \kappa_1$ non-zero elements below b_2 . Let $\kappa_2 < \kappa_1$ be the number of such non-zero elements. That is, letting

$$A_2 := \{a \wedge b_2 : a \in A_1\},$$

we have $|A_2| = \kappa_2 < \kappa_1$. If $\kappa_2 \leq \omega$, we are done by similar reasons as before. If not, then we may continue the procedure. However, the procedure will eventually terminate. This is because if not, then we would have an infinite sequence of decreasing cardinals

$$\kappa_0 > \kappa_1 > \kappa_2 > \dots,$$

which is impossible. Thus, A_0 has only countably many non-zero elements below b_ω .

At this point, we have argued that below the maximal antichain B_ω , \mathbb{B} has the c.c.c. Now, it must be that \mathbb{B} is $(\omega_1, 2)$ -distributive. Let us explain. It suffices to show that \mathbb{B} is $(\omega_1, 2)$ -distributive below each element of B_ω . Fix any $b_\omega \in B_\omega$. Below b_ω , \mathbb{B} is c.c.c. and $(\omega, 2)$ -distributive. Suppose, towards a contradiction, that \mathbb{B} is not $(\omega_1, 2)$ -distributive. Quoting a result of Baumgartner¹, there exists a Suslin tree which, when turned upside down, can be embedded into \mathbb{B} below b_ω . However, we assumed there are no Suslin trees. This completes the proof. \square

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¹This was discovered independently by Andreas Blass who was told it was already proved by James Baumgartner. However, neither the author nor Blass have been able to find a proof in the literature.